Part I - Normally-Distributed Asset Values

Gary Schurman, MBE, CFA

September, 2018

In Part I of this sereies we will build an asset value model that assumes that asset values are normally-distributed. To that end we will use the following hypothetical problem...

Our Hypothetical Problem

We are tasked with building an asset value model that assumes that asset values are normally-distributed. We are given the following go-forward model parameters...

Table 1: Portfolio Composition

Description	Value
Asset value at time zero (in dollars)	100.00
Annual return mean	12.00%
Annual distribution rate	5.00%
Annual return volatility	30.00%

Our task is to answer the following questions:

Question 1: What is expected asset value at the end of year three?

Question 2: What is the expected return mean and variance at the end of year three?

Question 3: What is the probability that asset value will be less than $100.00 at the end of year three?

Question 4: What is the probability that asset value will be negative at the end of year three?

Asset Value

We will define the variable $A(0)$ to be asset value at time zero, the variable m to be the expected rate of return, which is the the expected annual rate of return minus the expected annual distribution rate, and the variable v to be return variance. Using the parameters from Table 1 above the equations for these variables are...

$$A(0)=100.00 \text{ ...and... } m=0.1200-0.0500=0.0700 \text{ ...and... } v=0.3000^2=0.0900 \tag{1}$$

We will define the variable $A(t)$ to be random asset value at time t and the variable $W(t)$ to be the driving Brownian motion at time t. Using Equation (1) above the equation for random asset value is...

$$A(t)=A(0)\left(1+m\right)^t\left(1+\sqrt{v}\,W(t)\right) \text{ ...where... } W(t)\sim N\Bigg[0,t\Bigg] \tag{2}$$

We will define the variable Z to be a normally-distributed random variable with mean zero and variance one. We will standardize the random variable in Equation (2) above as follows...

$$Z=\frac{W(t)-0}{\sqrt{t}} \text{ ...such that... } W(t)=Z\sqrt{t} \text{ ...where... } Z\sim N\Bigg[0,1\Bigg] \tag{3}$$

Using Equation (3) above we will rewrite Equation (2) above as...

$$A(t)=A(0)\left(1+m\right)^t\left(1+\sqrt{v\,t}\,Z\right) \text{ ...where... } Z\sim N\Bigg[0,1\Bigg] \tag{4}$$

Using Appendix Equation (26) below the equation for the first moment of the distribution of random asset value at time t is...

$$\mathbb{E}\bigg[A(t)\bigg]=A(0)\bigg(1+m\bigg)^{t} \tag{5}$$

Using Appendix Equation (27) below the equation for the second moment of the distribution of random asset value at time t is...

$$\mathbb{E}\bigg[A(t)^2\bigg]=A(0)^2\bigg(1+m\bigg)^{2\,t}\bigg(1+v\,t\bigg) \tag{6}$$

Using Equation (5) above the equation for the mean of random asset value at time t is...

$$\text{mean of } A(t)=\mathbb{E}\bigg[A(t)\bigg]=A(0)\bigg(1+m\bigg)^{t} \tag{7}$$

Using Equations (5) and (6) above the equation for the variance of random asset value at time t is...

$$\text{variance of } A(t)=\mathbb{E}\bigg[A(t)^2\bigg]-\bigg[\mathbb{E}\bigg[A(t)\bigg]\bigg]^2=A(0)^2\bigg(1+m\bigg)^{2\,t}v\,t \tag{8}$$

To calculate the probability that asset value at time t will be above or below some threshold value we will solve asset value Equation (4) above in terms of the random variable Z. Using Appendix Equation (28) below the equation for Z is...

$$Z=\bigg[\frac{A_t}{A_0}\bigg(1+m\bigg)^{-t}-1\bigg]\bigg/\sqrt{v\,t}\ \text{...where...}\ Z\sim N\bigg[0,1\bigg] \tag{9}$$

Using Equation (**??**) above the equation for the probability of pulling a random variable from a standardized normal distribution where the value of the random variable is less than Z is...

$$\text{Prob}\bigg[Z\bigg]=\text{NORMSDIST(Z)}\quad\text{Note: NORMSDIST(value) is an Excel function} \tag{10}$$

Asset Return

We will define the variable $r(t)$ to be the random return over the time interval $[0,t]$. Using asset value Equation (4) above the equation for random asset return is...

$$r(t)=\frac{A(t)}{A(0)}-1 \tag{11}$$

Using Appendix Equation (29) below the equation for the first moment of the distribution of random asset return over the time interval $[0,t]$ is...

$$\mathbb{E}\bigg[r(t)\bigg]=\bigg(1+m\bigg)^{t}-1 \tag{12}$$

Using Appendix Equation (30) below the equation for the second moment of the distribution of random asset return over the time interval $[0,t]$ is...

$$\mathbb{E}\bigg[r(t)^2\bigg]=\bigg(1+m\bigg)^{2\,t}\bigg(1+v\,t\bigg)-2\bigg(1+m\bigg)^{t}+1 \tag{13}$$

Using Equation (12) above the mean of the distribution of random return over the time interval $[0,t]$ is...

$$\text{mean of } r(t)=\mathbb{E}\bigg[r(t)\bigg]=\bigg(1+m\bigg)^{t}-1 \tag{14}$$

Using Equations (12) and (13) above above the variance of the distribution of random return over the time interval $[0,t]$ is...

$$\text{variance of } r(t)=\mathbb{E}\bigg[r(t)^2\bigg]-\bigg[\mathbb{E}\bigg[r(t)\bigg]\bigg]^2=\bigg(1+m\bigg)^{2\,t}v\,t \tag{15}$$

The Answers To Our Hypothetical Problem

Question 1: What is expected asset value at the end of year three?

Using Equations (1) and (7) above expected asset value at the end of year three is...

$$A(3) = 100.00 \times \left(1 + 0.0700\right)^3 = 122.50 \tag{16}$$

Question 2: What is the expected return mean and variance at the end of year three?

Using Equation (14) above the return mean is...

$$\text{mean} = \left(1 + 0.0700\right)^3 = 1.2250 \tag{17}$$

Using Equation (15) above the return variance is...

$$\text{variance} = \left(1 + 0.0700\right)^{2 \times 3} \times 0.0900 \times 3 = 0.4052 \tag{18}$$

Question 3: What is the probability that asset value will be less than \$100.00 at the end of year three?

Using Equations (1) and (9) above the value of the random variable Z is...

$$Z = \left[\frac{100.00}{100.00} \times \left(1 + 0.0700\right)^{-3} - 1\right] \div \sqrt{0.0900 \times 3} = -0.3535 \tag{19}$$

Using Equation (10) above the probability is...

$$\text{Prob}\left[-0.3535\right] = \text{NORMSDIST(-0.3535)} = 0.3618 \tag{20}$$

Question 4: What is the probability that asset value will be negative at the end of year three?

Using Equations (1) and (9) above the value of the random variable Z is...

$$Z = \left[\frac{0.00}{100.00} \times \left(1 + 0.0700\right)^{-3} - 1\right] \div \sqrt{0.0900 \times 3} = -1.9246 \tag{21}$$

Using Equation (10) above the probability is...

$$\text{Prob}\left[-1.9246\right] = \text{NORMSDIST(-1.9246)} = 0.0271 \tag{22}$$

Appendix

A. Using the definition of the normally-distributed random variable Z in Equation (3) above note the following...

$$\mathbb{E}\left[Z\right] = 0 \text{ ...and... } \mathbb{E}\left[Z^2\right] = 1 \tag{23}$$

B. Using Appendix Equation (23) above the solution to the following expectation is...

$$\begin{aligned}\mathbb{E}\left[1 + \sqrt{v\,t}\,Z\right] &= 1 + \sqrt{v\,t}\,\mathbb{E}\left[Z\right] \\ &= 1 \end{aligned} \tag{24}$$

C. Using Appendix Equation (23) above the solution to the following expectation is...

$$\begin{aligned}\mathbb{E}\left[\left(1+\sqrt{v\,t}\,Z\right)^2\right] &= \mathbb{E}\left[1+2\sqrt{v\,t}\,Z+v\,t\,Z^2\right] \\ &= 1+2\sqrt{v\,t}\,\mathbb{E}\left[Z\right]+v\,t\,\mathbb{E}\left[Z^2\right] \\ &= 1+v\,t \end{aligned} \tag{25}$$

D. Using Equations (4) and (24) above the equation for the first moment of the distribution of random asset value at time t is...

$$\begin{aligned}\mathbb{E}\left[A(t)\right] &= \mathbb{E}\left[A(0)\left(1+m\right)^t\left(1+\sqrt{v\,t}\,Z\right)\right] \\ &= A(0)\left(1+m\right)^t\mathbb{E}\left[1+\sqrt{v\,t}\,Z\right] \\ &= A(0)\left(1+m\right)^t \end{aligned} \tag{26}$$

E. Using Equations (4) and (25) above the equation for the second moment of the distribution of random asset price at time t is...

$$\begin{aligned}\mathbb{E}\left[A(t)^2\right] &= \mathbb{E}\left[A(0)^2\left(1+m\right)^{2\,t}\left(1+\sqrt{v\,t}\,Z\right)^2\right] \\ &= A(0)^2\left(1+m\right)^{2\,t}\mathbb{E}\left[\left(1+\sqrt{v\,t}\,Z\right)^2\right] \\ &= A(0)^2\left(1+m\right)^{2\,t}\left(1+v\,t\right) \end{aligned} \tag{27}$$

F. Using asset price Equation (4) above and solving for the random variable Z...

$$\begin{aligned}A(t) &= A(0)\left(1+m\right)^t\left(1+\sqrt{v\,t}\,Z\right) \\ \frac{A_t}{A_0}\left(1+m\right)^{-t} &= 1+\sqrt{v\,t}\,Z \\ Z &= \left[\frac{A_t}{A_0}\left(1+m\right)^{-t}-1\right]\Big/\sqrt{v\,t} \end{aligned} \tag{28}$$

G. Using Equations (11) and (26) above the equation for the first moment of the distribution of the random rate of return over the time interval $[0,t]$ is...

$$\begin{aligned}\mathbb{E}\left[r(t)\right] &= \mathbb{E}\left[\frac{A(t)}{A(0)}-1\right] \\ &= \frac{1}{A(0)}\,\mathbb{E}\left[A(t)\right]-1 \\ &= \left(1+m\right)^t-1 \end{aligned} \tag{29}$$

H. Using Equations (11), (26) and (27) above the equation for the second moment of the distribution of the random rate of return over the time interval $[0,t]$ is...

$$\begin{aligned}\mathbb{E}\left[r(t)^2\right] &= \mathbb{E}\left[\left(\frac{A(t)}{A(0)}-1\right)^2\right]\\ &= \mathbb{E}\left[\frac{A(t)^2}{A(0)^2}-2\,\frac{A(t)}{A(0)}+1\right]\\ &= \frac{1}{A(0)^2}\,\mathbb{E}\left[A(t)^2\right]-\frac{2}{A(0)}\,\mathbb{E}\left[A(t)\right]+1\\ &= \left(1+m\right)^{2\,t}\left(1+v\,t\right)-2\left(1+m\right)^{t}+1 \end{aligned} \tag{30}$$